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ONE PLASTICITY MODEL FOR PROBLEMS OF PLASTIC METAL WORKING

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Scalar and tensor models of plastic flow of metals extending plasticity theory are considered over a wide range of temperatures and strain rates. Equations are derived using the physico-phenomenological approach based on modern concepts and methods of the physics and mechanics of plastic deformation. For hardening and viscoplastic solids, a new mathematical formulation of the boundary-value plasticity problem taking into account loading history is obtained. Results of testing of the model are given. A numerical finite-element algorithm for the solution of applied problems is described.

Key words: *plastic strain, plasticity theory, loading history, physico-phenomenological approach, hardening solid, viscoplasticity.*

Introduction. In the theory of plastic metal working (PMW), mechanicomathematical models based on the mathematical theory of plasticity have been developed more fully as opposed to physical models [1]. At present, these models are used in all calculations and studies of technological PMW processes. However, the gap between theory and practice (modern technology) needs becomes more and more obvious [2, 3].

One of the effective PMW technologies is multitransition cold forging process, which is finding increasingly wide application. The cold forging technology has been developed on the basis of production experience since, in engineering design, plasticity theory does not allow one to take into account the deformation anisotropy of the properties of the material worked and, hence, complex loading.

The hot forging process, which is the most widely used type of PMW, is studied to increase the forging accuracy, to reduce the material and power consumption, and to make possible the working of new difficult-to-form high alloys with special functional properties. The mathematical theory of plasticity provides only approximate calculations of the strain–stress state of workpieces in hot volumetric forging since it ignores the loading history involving time dependences of strain, strain rate, and temperature. This is due to the fact that the behavior of viscoelastic viscoplastic solids is not described by the phenomenological theory of plasticity [1, 4].

Isothermal forging under various conditions, including the superplasticity conditions of the material worked, is gradually finding technological applications. However, calculations and studies of these technological processes using the classical theory of plasticity involve difficulties since under superplastic deformation conditions in the optimum range of rates, materials have the so-called descending strain diagram. This diagram is typical of many high alloys subjected to hot deformation. These materials does not obey the Drucker postulate, which implies the maximum principles underlying plasticity theory [4]. Solution of the above-mentioned problems using the phenomenological approach involves significant difficulties.

In recent decades, there has been significant progress in the physics of strength and plasticity, which is the basis for the physical direction in PMW theory [5]. One direction for the further development of plasticity theory and, hence, PMW theory, is the synthesis of ideas of the mechanics and physics of plastic strain as two fundamental disciplines having the same research obbject — plastic deformation and failure of materials. Recently, this direction has split into two approaches.

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The first approach, developed since the late 1940s [6], has led to the creation of a new version of plasticity theory — slip theory [2, 7], resulting from a revision of the mathematical theory of plasticity taking into account modern advances in the physics of strength and plasticity. Slip theory has been continuously improved [3], but it has not yet found application in designing plastic metal forming processes. Obviously, this approach underlies the currently developed models for elastoplastic deformations of metals [8, 9]: the geometrical model of the defect structure of elastoplastic continuous media, the gauge theories of defects, and the generalized thermodynamic model.

The second approach is based on the synthesis of methods and basic propositions of the mathematical theory of plasticity and the physics of strength and plasticity. Obviously, this approach is similar to the synthesis of phenomenological thermodynamics and molecular-kinetic theory into statistical thermodynamics [10]. In this approach, the physics of plastic deformation should give a basis for the derivation of the laws of deformation of metals in various structural states and under various thermomechanical conditions for uniaxial stress states. Because this approach is physical and, hence, clarifies the mechanisms of deformation processes, it makes it possible to justify the choice of the determining parameters and the form of the loading function (plasticity conditions), the formulation of the corresponding maximum principle, the associated flow rule, and governing relations. In the mathematical theory of plasticity, these problems are solved by adopting a certain hypothesis on the plastic behavior of the material. This is responsible for the difficulty in solving the above-mentioned problems.

This paper is devoted to generalization and development of the basic theoretical results obtained using the second approach. Solutions of some practical problems based on the proposed theory are given in [11, 12].

1. Scalar Models of Plastic Deformation (Uniaxial Stress State). From a thermodynamic point of view, plastic deformation is a nonequilibrium process, and it is therefore can be described theoretically within the framework of the microstructural approach using kinetic methods [10].

Based on the modern concepts of strength and plasticity physics about the thermally activated micromechanisms of plastic deformation, hardening, and softening and using the kinetic equations of dislocation density balance, the scalar flow law for metals (ignoring elastic deformation) was obtained in [13, 14]:

$$\begin{gathered}
\sigma_{(g)} = \alpha(T) m G(T) b\Big(1 - \frac{2.5kT}{G(T)b^3}\ln\frac{\dot\varepsilon_{0(g)}}{\dot\varepsilon_{(g)}}\Big)\sqrt{\rho_{s(g)}}, \qquad \rho_{s(g)} = \rho_{s(g-1)} + d\rho_{s(g)},\\
d\rho_{s(g)} = \Big[\frac{1}{b\lambda} - \frac{\rho_{s(g-1)}}{\dot\varepsilon_{(g)}}\,\nu_{\rm D} b\sqrt{\rho_{s(g-1)}}\,\exp\Big(-\frac{G(T)b^3 - 1.443\sigma_{(g-1)}b^2/\sqrt{\rho_{s(g-1)}}}{2.5kT}\Big)\Big]d\varepsilon_{(g)},\\
d\sigma_{(g)} = \Big[\frac{\alpha(T)mG(T)b}{2\sqrt{\rho_{s(g)}}} - \frac{mkT_{(g)}}{2b^2\sqrt{\rho_{s(g)}}}\Big(1 + \ln\frac{\dot\varepsilon_* b\sqrt{\rho_{s(g)}}}{\dot\varepsilon_{(g)}}\Big)\Big]\,d\rho_{s(g)},\\
\varepsilon_{(g)} = \varepsilon_{(g-1)} + d\varepsilon_{(g)}, \quad d\varepsilon_{(g)} = \Big(\frac{2}{3}\,d\varepsilon_{ij(g)}\,d\varepsilon_{ij(g)}\Big)^{1/2}, \quad \dot\varepsilon_{(g)} = \frac{d\varepsilon_{(g)}}{dt_{(g)}}, \quad dt_{(g)} = \frac{dU_{z(g)}}{v_{(g)}}.
\end{gathered} \tag{1}$$

Here $\sigma_{(g)}$ and $d\sigma_{(g)}$ are the stress intensity and the stress intensity increment in the loading step $g = 1, 2, \ldots, n$ in step-by-step calculation of plastic flow, α is the interdislocation interaction parameter, $m = 3.1$ is the Taylor factor, G is the shear modulus, b is the dislocation Burgers vector modulus averaged over the slip system [for metals, $b = (2\text{–}3)\cdot 10^{-8}$ cm], k is Boltzmann constant, T is the thermodynamic temperature, $\dot\varepsilon_0 = \dot\varepsilon_* b\sqrt{\rho_s}$, $\dot\varepsilon_* \simeq 9\cdot 10^6$ sec^{-1}, $\rho_{s(g)}$ and $d\rho_{s(g)}$ are the volume-averaged scalar density and scalar density increment of immobile dislocations, λ is the mean free path of mobile dislocations, $\dot\varepsilon_{(g)}$ is the plastic strain rate intensity, $\nu_{\rm D} = 10^{12}\text{–}10^{13}$sec^{-1} is the frequency of thermal fluctuations of ions in the crystal lattice (Debye frequency), $d\varepsilon_{(g)}$ is the plastic strain increment rate, $d\varepsilon_{ij(g)}$ is the increment of the strain tensor components, $dt_{(g)}$ is the increment of the deformation time in loading step g, $dU_{z(g)}$ is the displacement increment of the deforming tool, and $v_{(g)}$ is the linear rate of displacement of the deforming tool in loading step g.

Because the theory being developed is initially focused on the solution of practical problems [13, 14], i.e., it is designed for practical use, model (1) ignores some details of the dislocation plasticity mechanism, for example, the occurrence of unlocal relations between dislocation density and strain [15].

For a rigid-viscoplastic solid, the initial (loading step $g = 1$) yield point of the material is defined by the formula

$$\sigma_{\rm y} = \alpha(T)mG(T)b\Big(1 - \frac{2.5kT}{G(T)b^3}\ln\frac{\dot\varepsilon_{0(1)}}{\dot\varepsilon_{(1)}}\Big)\sqrt{\rho_{s0}}, \tag{2}$$

where ρ_{s0} is the initial (before heating and deformation) density of immobile dislocations in the material.

In the current loading step g, the initial (instantaneous) yield point of the material is given by

$$\sigma^{\mathrm{t}}_{(g)} = \alpha(T)mG(T)b\Big(1 - \frac{2.5kT}{G(T)b^3}\ln\frac{\dot\varepsilon_{0(g)}}{\dot\varepsilon_{(g)}}\Big)\sqrt{\rho_{s(g-1)}}. \tag{3}$$

Operator (1) describes the plastic deformation of metals over a wide range of temperatures (cold, warm, and hot deformations) and strain rates in which the dominating mechanism is dislocation slip in grains. Except for special types of working processes, dependence (1) is valid for the range of temperatures and rates used in plastic metal working in industry. In each loading step g, the operator (1) assigns a value of σ to the functions $\dot\varepsilon(\varepsilon)$, $T(\varepsilon)$, $\rho_s(\varepsilon)$, and $\varepsilon(t)$, i.e., it takes into account the loading history determined by the dependences of $\dot\varepsilon$ and T on deformation time.

In cold deformation (except in high-rate deformation), the stress σ depends little on the strain rate $\dot\varepsilon$ and dislocations overcome barriers by force action. The cold deformation conditions [14] are written as

$$\dot\varepsilon_0 = \dot\varepsilon, \qquad \dot\varepsilon = \nu_{\mathrm{D}} b\sqrt{\rho_s}\exp\Big(-\frac{Gb^3 - 1.443\sigma b^2/\sqrt{\rho_s}}{2.5kT}\Big). \tag{4}$$

In view of (4), relation (1) leads to the scalar flow law for a rigid plastic hardening solid (cold deformation of metals) in differential form

$$\begin{gathered}\sigma_{(g)} = \alpha mGb\sqrt{\rho_{s(g)}}, \qquad \rho_{s(g)} = \rho_{s(g-1)} + d\rho_{s(g)},\\ d\rho_{s(g)} = \Big(\frac{1}{b\lambda} - \rho_{s(g-1)}\Big)d\varepsilon_{(g)}, \quad \varepsilon_{(g)} = \varepsilon_{(g-1)} + d\varepsilon_{(g)}, \quad d\varepsilon_{(g)} = \Big(\frac{2}{3}\,d\varepsilon_{ij(g)}\,d\varepsilon_{ij(g)}\Big)^{1/2}\end{gathered} \tag{5}$$

and in integral form

$$\sigma = \alpha mGb\Big(\frac{(b\lambda)^{-1}[\exp(\varepsilon) - 1] + \rho_{s0}}{\exp(\varepsilon)}\Big)^{1/2}. \tag{6}$$

From (2) subject to (4) or from (6) for $\varepsilon = 0$, we obtain the following expression for the initial yield point of the material:

$$\sigma_{\mathrm{y}} = \alpha mGb\sqrt{\rho_{s0}}. \tag{7}$$

The values of the parameters ρ_{s0} and λ, which are required to calculate strain diagrams by formulas (1) and (5) and are characteristics of material structure, can be determined using metallographic analysis methods. However, they are easier to determine from the formulas

$$\rho_{s0} = \frac{(\sigma^{\exp}_{\mathrm{y}})^2}{(\alpha mGb)^2}, \qquad \lambda = \frac{b(\alpha mG)^2[\exp(\varepsilon) - 1]}{\sigma^2\exp(\varepsilon) - (\alpha mGb)^2\rho_{s0}} \tag{8}$$

using experimental strain diagrams $\sigma(\varepsilon)$ obtained in standard compression tests under cold deformation conditions. In (8), $\sigma^{\exp}_{\mathrm{y}}$ is the experimental yield point for cold deformation; $\varepsilon \in [0.4; 0.6]$, and σ is the strain intensity and its corresponding stress intensity in the experimental strain diagram. Formulas (8) follow from Eqs. (7) and (6).

For a group of steels, it is found that the temperature dependences of the shear modulus G and the interdislocation interaction parameter α are identical and, with accuracy sufficient for engineering design, they are approximated by the relations

$$G(T) = (-65.94\cdot 10^{-3}T^2 + 41.08T + 77.82\cdot 10^3)G_m/84{,}000,$$

$$\alpha(T) = -1.625\cdot 10^{-7}T^2 + 10.123\cdot 10^{-5}T + 0.194,$$

where G_m [MPa] is the shear modulus of the material at a temperature of 293 K.

In the case of steels at the temperatures corresponding to hot deformation, the best approximation is obtained under the assumption of steps in the dependences $\alpha(T)$ and $G(T)$ at the polymorphic transformation temperature $\mathrm{Fe}_\alpha \to \mathrm{Fe}_\gamma$, i.e., at $T \geqslant 1000$ K:

$$\alpha(T) = -1.625\cdot 10^{-7}T^2 + 10.123\cdot 10^{-5}T + 0.121, \quad G(T) = (-65.94\cdot 10^{-3}T^2 + 41.08T + 89.38\cdot 10^3)G_m/84{,}000.$$

We note that the stepwise increase in the elastic moduli E and G in the $\mathrm{Fe}_\alpha \to \mathrm{Fe}_\gamma$ transformation is known (see [5]).

More detailed (compared to [13, 14] studies of model (1) using experimental strain diagrams for steels have shown that this model allows one to determine not only cold deformation conditions [14] but also warm and hot deformation conditions. It has been found that warm deformation occurs with a change (decrease with increasing strain and density ρ_s) in the activation volume $V = b^2/\sqrt{\rho_s}$, i.e., according to the third equation of system (1). Hot deformation is performed at constant activation volume ($V = b^2/\sqrt{\rho_{s0}} = \text{const}$); in this case, the third equation in (1) is written as

$$d\rho_{s(g)} = \Big[\frac{1}{b\lambda} - \frac{\rho_{s(g-1)}}{\dot{\varepsilon}_{(g)}}\,\nu_{\rm D} b\sqrt{\rho_{s(g-1)}}\,\exp\Big(-\frac{G(T)b^3 - 1.443\sigma_{(g-1)}b^2/\sqrt{\rho_{s0}}}{2.5kT}\Big)\Big]d\varepsilon_{(g)}.$$

2. Tensor Models of Plastic Strain (Overall Stress State). It has been shown [16] that scalar models adequately describe the plastic deformation of metals over a wide range of temperatures and strain rates. To take into account the loading history due to the occurrence and development of deformation anisotropy, i.e., to take into account the influence of the trajectory loading in the design and mathematical simulation of cold forging operations and processes, it is necessary to use the governing relations for an isotropic material with anisotropic hardening [11]:

$$d\varepsilon_{ij} = \frac{3}{2}\,\frac{d\varepsilon}{\Phi(\varepsilon)}\,[s_{ij} - g(\varepsilon)\varepsilon_{ij}]. \tag{9}$$

Because in the proposed plasticity models, the scalar functions of the accumulated plastic strain $\Phi(\varepsilon)$ and $g(\varepsilon)$ in (9) are defined by the relations

$$\Phi(\varepsilon) = \frac{\alpha m G b}{2}\Big[\Big(\frac{(b\lambda)^{-1}[\exp(\varepsilon) - 1] + \rho_{s0}}{\exp(\varepsilon)}\Big)^{1/2} + (\rho_{s0} + A\varepsilon)^{1/2}\Big]; \tag{10}$$

$$g(\varepsilon) = \frac{\alpha m G b}{3\varepsilon}\Big[\Big(\frac{(b\lambda)^{-1}[\exp(\varepsilon) - 1] + \rho_{s0}}{\exp(\varepsilon)}\Big)^{1/2} - (\rho_{s0} + A\varepsilon)^{1/2}\Big], \tag{11}$$

this model can be used to solve practical problems.

To determine the quantity A in Eqs. (10) and (11), it is necessary to perform an experiment including two stages: 1) deformation of cylindrical workpieces under conditions of simple stretching (by drawing or pressing) with an average degree of strain $0.25 \leqslant \varepsilon^+ \leqslant 0.70$; 2) upsetting of cylindrical samples cut from the rod with determination of the compressive yield point $\sigma_{\rm y}^s$. The quantity A is calculated by the formula [11]

$$A = \frac{(\sigma_{\rm y}^s)^2 - (\alpha m G b)^2\rho_{s0}}{(\alpha m G b)^2\varepsilon^+}.$$

Solution of test and technological problems of cold forging based on the proposed plasticity model (5), (6), (9)–(11) using a finite-element method shows that compared to the phenomenological theory of a hardening plastic solid, this model provides higher accuracy in calculations of strain–stress states and deformation forces for forging due to accounting for loading history [12].

At elevated temperatures (warm and hot deformations), metals show viscoplastic viscoelastic properties. The well-known tensor laws of deformation for viscoplastic viscoelastic solids do not consider the loading history due to dependences $T(t)$ and $\dot{\varepsilon}(t)$ [4], and, therefore, the physicomathematical scalar flow law (1), (2) should be extended to the overall stress state.

The extension is performed using the plasticity method based on the choice of the loading function, which is traditional in the mathematical theory [16]. The difference is that the loading function is not taken as a hypothesis but is a consequence of the scalar physical flow law.

From the first equation in (1), Eqs. (2) and (3), and the definition of the yield point, it follows that a viscoplastic solid has no yield point. The initial and current yield stresses are determined by the strain rate and temperature. Consequently, for the viscoplastic solids considered, unloading is absent. In addition to the neutral loading process for viscoplastic solids, thermal and dynamic recovery processes, and, accordingly, thermal and dynamic (high-rate) hardening are also determined in terms of mechanics [16].

From the first four equations of system (1) and Eqs. (3), it follows that, in each step g, the yield stress can be represented as

$$\sigma_{(g)} = \sigma^{\rm t}_{(g)} + d\sigma_{(g)},$$

where $\sigma^{\rm t}_{(g)}$ is the initial yield stress in step g for the values of $\dot{\varepsilon}_{(g)}$ and $T_{(g)}$ valid for this step; $d\sigma_{(g)}$ is the stress increment in the loading step g due to the increment $d\varepsilon_{(g)}$ leading to the increment $d\rho_{s(g)}$. Based on this scalar physical flow law for viscoplastic solids, the loading function and the instantaneous Huber–Mises yield condition which are instantaneous in the deformation step are expressed as

$$f_{(g)}\Big[(\sigma^{\rm t}_{ij(g)} + d\sigma_{ij(g)}), \Phi_{(g)}\Big(\sum_1^{g-1} d\varepsilon_{(g)} + d\varepsilon_{(g)}\Big)\Big] = 0,$$

$$f_{(g)}(\sigma^{\rm t}_{ij(g)} + d\sigma_{ij(g)}) = \frac{3}{2}\,(s^{\rm t}_{ij(g)} + ds_{ij(g)})(s^{\rm t}_{ij(g)} + ds_{ij(g)}) - \Big[\Phi_{(g)}\Big(\sum_1^{g-1} d\varepsilon_{(g)} + d\varepsilon_{(g)}\Big)\Big]^2 = 0,$$

where $s^{\rm t}_{ij(g)}$ and $ds_{ij(g)}$ are the deviators of the tensors $\sigma ij(g)^{\rm t}$ and $d\sigma_{ij(g)}$.

For viscoplastic solids, the new maximum principle has the form [16]

$$(\sigma^{\rm t}_{ij} + d\sigma_{ij})_{(g)} d\varepsilon_{ij(g)} > (\sigma^{\rm t0}_{ij} + d\sigma^{0}_{ij})_{(g)} d\varepsilon_{ij(g)}$$

provided that $f^0_{(g)}(\dot{\varepsilon}^0_{(g)}, T^0_{(g)}) < f_{(g)}(\dot{\varepsilon}_{(g)}, T_{(g)})$, $\dot{\varepsilon}^0_{(g)} < \dot{\varepsilon}_{(g)}$, and $T^0_{(g)} > T_{(g)}$.

The associated flow rule in the loading step g has the form

$$\frac{\partial}{\partial\,(\sigma^{\rm t}_{ij} + d\sigma_{ij})_{(g)}}\,[(\sigma^{\rm t}_{ij} + d\sigma_{ij})_{(g)}\,d\varepsilon_{ij(g)} - (d\lambda)_{(g)} f_{(g)}] = 0,$$

where $d\lambda$ is the Lagrangian multiplier.

The governing relations for viscoplastic solids become

$$d\varepsilon_{ij(g)} = \frac{3}{2}\,\frac{d\varepsilon_{(g)}}{(\sigma^{\rm t} + d\sigma)_{(g)}}\,(s^{\rm t}_{ij} + ds_{ij})_{(g)}, \tag{12}$$

where the quantities $\sigma^{\rm t}$ and $d\sigma$ in each loading step are calculated by relation (3) and the fourth equation in system (1), respectively.

The tensor law (12), which is an extension of flow theory, leads to the governing relations for hardening plastic and nonlinear viscous solids, which are special cases of (12). In [16], it is shown that viscoplastic solids obey the incompressibility condition. Hence, in (12), the strain increment tensor is a deviator.

Expression (12) can be written as

$$(s^{\rm t}_{ij} + ds_{ij})_{(g)} = \frac{2}{3}\,\frac{\sigma^{\rm t}_{(g)}}{d\varepsilon_{(g)}}\,d\varepsilon_{ij(g)} + \frac{2}{3}\,\frac{d\sigma_{(g)}}{d\varepsilon_{(g)}}\,d\varepsilon_{ij(g)}. \tag{13}$$

Equation (13) leads to

$$d\varepsilon_{ij(g)} = \frac{3}{2}\,\frac{d\varepsilon_{(g)}}{\sigma^{\rm t}_{(g)}}\,s^{\rm t}_{ij(g)}, \qquad d\varepsilon_{ij(g)} = \frac{2}{3}\,\frac{d\varepsilon_{(g)}}{d\sigma_{(g)}}\,ds_{ij(g)}. \tag{14}$$

In contrast to (12), Eqs. (14), which indicate that the directing tensors $\bar{\sigma}^{\rm t}_{ij(g)}$ and $d\bar{\sigma}_{ij(g)}$ are equal, are conveniently used to solve practical problems.

The scalar law (1) describes plastic deformation under conditions of time-varying strain rate and temperature adequately to experimental data [16]. The extension of this law to the overall strain–stress state using the mathematical theory of plasticity was performed without adopting new hypotheses. This gives reason to believe that Eq. (12) adequately describes the plastic deformation of viscoplastic solids for complex stress states.

Using the developed model of plasticity, it is possible to give a new mathematical formulation of the boundary-value problem. For the analysis of cold deformation processes (hardening solids), the system of equations of this problem includes:

1) the differential equilibrium equations in stress increments:

$$d\sigma_{ij,j} = 0; \tag{15}$$

2) the geometrical Cauchy relations in strain and displacement increments:

$$d\varepsilon_{ij} = (du_{i,j} + du_{j,i})/2; \tag{16}$$

3) governing relations (8)–(10).

For the of analysis of warm and hot deformations (viscoplastic solids), the system of equations of the boundary-value problem includes Eqs. (15) and (16) and the governing relations (14) and (1)–(3). Both systems are supplemented by boundary-value conditions corresponding to the particular problem.

3. Linearization of Governing Relations and General Algorithm for Solving Boundary-Value Problems. In view of the specificity of the theory developed [to track the behavior of each material particle, all equations are written in differential (incremental) form], in solution of practical problems, for example, in mathematical simulation of technological operations and PMW processes for the purpose of rationalizing and optimizing technological parameters, it is reasonable to solve the formulated boundary-value problem using a numerical finite-element method that allows implementing the necessary step-by-step computation algorithm. (The solution method and algorithm are well-elaborated [17].) In each loading step g, a quasi-elastic problem is solved using the known linear equations of elasticity to which the second equation (14) is transformed if the quasi-elastic constants in the step g are defined as follows: $E^*_{(g)} = d\sigma_{(g)}/d\varepsilon_{(g)}$ is the instantaneous secant modulus; $\nu^*_{(g)} = 1/2 - (1-2\nu)E^*_{(g)}/(2E)$ is the instantaneous Poisson ration at small elastoplastic strains; $K^*_{(g)} = E^*_{(g)}/(1-2\nu^*_{(g)})$ is the instantaneous overall compression quasimodulus [ν and E are elastic constants of the material (Poisson ratio and longitudinal elasticity modulus, respectively)].

Once $d\varepsilon_{ij(g)}$ and $d\sigma_{ij(g)}$ are determined, the tensor components $\sigma^{\rm t}_{ij(g)}$ are found from the first equation in (14). In the loading step g, we have the stress state $\sigma_{ij(g)} = \sigma^{\rm t}_{ij(g)} + d\sigma_{ij(g)}$.

To solve practical problems of cold plastic deformation, for example, to calculate and analyze the technological processes of cold forging of metals, it is necessary to linearize the governing relations (9). In this case, the presence of the scalar physical models (10) and (11) considerably simplifies this procedure in the numerical implementation of the corresponding boundary-value problem. Similarly to (12) we write Eq. (9) as

$$d\varepsilon_{ij(g)} = \frac{3}{2}\,\frac{d\varepsilon_{(g)}}{\Phi_{(g-1)} + d\Phi_{(g)}}\,[s_{ij(g-1)} + ds_{ij(g)} - g_{(g-1)}\varepsilon_{ij(g-1)} - dg_{(g)}\varepsilon_{ij(g)} - g_{(g)}\,d\varepsilon_{ij(g)}]. \qquad (17)$$

As in the case of Eq. (12), Eq. (17) leads to the relations

$$d\varepsilon_{ij(g)} = \frac{3}{2}\,\frac{d\varepsilon_{(g)}}{\Phi_{(g-1)}}\,[s_{ij(g-1)} - g_{(g-1)}\varepsilon_{ij(g-1)}];$$

$$d\varepsilon_{ij(g)} = \frac{3}{2}\,\frac{d\varepsilon_{(g)}}{d\Phi_{(g)}}\,[ds_{ij(g)} - dg_{(g)}\varepsilon_{ij(g)} - g_{(g)}\,d\varepsilon_{ij(g)}]. \qquad (18)$$

According to the theory of plasticity of an isotropic material with anisotropic hardening [4], in Eq. (18) the expression in square brackets is the active stress increment deviator $ds_{ij(g)} - dg_{(g)}\varepsilon_{ij(g)} - g_{(g)}d\varepsilon_{ij(g)} = ds^a_{ij(g)}$, or

$$d\varepsilon_{ij(g)} = \frac{3}{2}\,\frac{d\varepsilon_{(g)}}{d\Phi_{(g)}}\,ds^a_{ij(g)}. \qquad (19)$$

With the use of the substitution of the values determined for each step and each material particle

$$E^{**}_{(g)} = \frac{d\Phi_{(g)}}{d\varepsilon_{(g)}} = \frac{1}{4}\,\alpha mGb\Big[\Big(\frac{(b\lambda)^{-1}[\exp{(\varepsilon_{(g)})} - 1] + \rho_{s0}}{\exp{(\varepsilon_{(g)})}}\Big)^{-1/2}\frac{(b\lambda)^{-1} - \rho_{s0}}{\exp{(\varepsilon_{(g)})}} + A(\rho_{s0} + A\varepsilon_{(g)})^{-1/2}\Big],$$

$$\nu^*_{(g)} = \nu^{**}_{(g)} = \frac{1}{2} - \frac{1-2\nu}{2E}\,E^{**}_{(g)}$$

in each loading step g, Eqs. (19) are reduced to the incremental linear equations of elasticity.

After the solution of the quasi-elastic problem in the loading step g using the finite-element method and determination of the quantities $d\varepsilon_{ij(g)}$, $\varepsilon_{ij(g)}$, $d\varepsilon_{(g)}$, $d\sigma^a_{ij(g)}$, and $d\sigma^a_{0(g)}$, the increments of the acting stress deviator components are calculated by the formulas $ds_{ij(g)} = ds^a_{ij(g)} + dg_{(g)}\,\varepsilon_{ij(g)} + g_{(g)}\,d\varepsilon_{ij(g)}$ where, according to (10), we have

$$dg_{(g)} = \frac{\alpha mGb}{3\varepsilon_{(g)}}\Big\{\frac{1}{2}\Big[\Big(\frac{(b\lambda)^{-1}[\exp{(\varepsilon_{(g)})} - 1] + \rho_{s0}}{\exp{(\varepsilon_{(g)})}}\Big)^{-1/2}\frac{(b\lambda)^{-1} - \rho_{s0}}{\exp{(\varepsilon_{(g)})}} - A(\rho_{s0} + A\varepsilon_{(g)})^{-1/2}\Big]$$
$$- \frac{1}{\varepsilon_{(g)}}\Big[\Big(\frac{(b\lambda)^{-1}[\exp{(\varepsilon_{(g)})} - 1] + \rho_{s0}}{\exp{(\varepsilon_{(g)})}}\Big)^{1/2} - (\rho_{s0} + A\varepsilon_{(g)})^{1/2}\Big]\Big\}d\varepsilon_{(g)}.$$

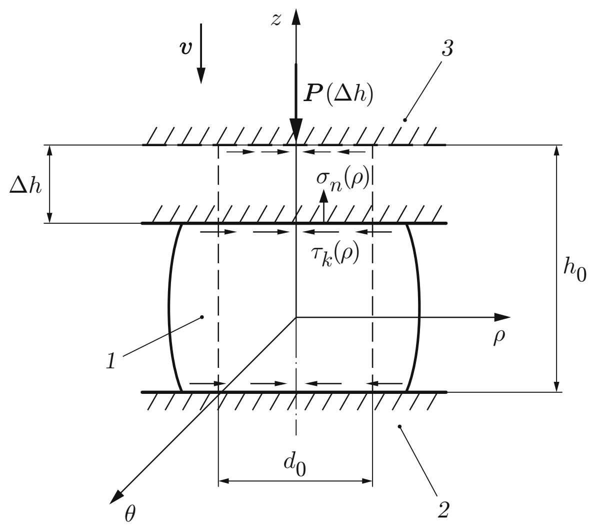

Fig. 1. Diagram of upsetting of a cylindrical workpiece by flat dies: workpiece (1), fixed die (2), and movable die (3); the initial and current states of the workpiece are shown by the dashed and solid lines, respectively.

Next, a calculation is made of the components of the stress increment tensor $d\sigma_{ij(g)} = ds_{ij(g)} + d\sigma_{0(g)}\,\delta_{ij}$, where $d\sigma_{0(g)} = K^{**}_{(g)}\,d\varepsilon_{0(g)}$, δ_{ij} is the Kronecker symbol, and $K^{**}_{(g)} = E^{**}_{(g)}/(1-2\nu^{**}_{(g)})$.

4. Testing of the Model. As noted in Sec.2, the model of a hardening plastic solid (which corresponds to the cold deformation of metals) was tested by solving technological plasticity problems (see, for example, [12]). The adequacy of the model of a viscoplastic solid (which corresponds to the warm and hot deformations of metals) was verified by solving the problem of upsetting of a cylindrical workpiece by flat dies under contact friction conditions.

Figure 1 shows a diagram of the upset forging. A cylindrical workpiece of St. 10 steel of initial diameter $d_0 = 6$ mm and height $h_0 = 7$ mm is upset by fixed and movable flat dies. The movable die moves at a rate $v = \text{const}$ (in the calculation, three displacement rates were specified: $v = 8.3 \cdot 10^{-1}$, $8.3 \cdot 10^{-2}$, and $8.3 \cdot 10^{-3}$ mm/sec). The contact surfaces of the workpiece with the dies (the ends of the cylinder) were lubricated with a suspension of colloidal graphite in mineral oil. This technological operation was performed on a 1231U-10 universal test machine, which allows the force necessary for workpiece deformation P to be recorded with an error not more than 2% of the measured quantity and the displacement of the movable die Δh. The machine is equipped with a heating furnace to perform deformation under isothermal conditions. The workpiece and die had the same temperature which was constant during the deformation and equal to 850°C. (For St. 10 steel, this temperature corresponds to the hot deformation temperature exceeding the recrystallization temperature.)

The adequacy of the hot deformation model was verified by comparing experimental and calculated dependences $P(\Delta h)$. The calculated dependences $P(\Delta h)$ were obtained by numerical computer simulation of the workpiece deformation during upsetting forging. The axisymmetric problem was solved using a finite-element method in displacement increments using the algorithm described in Sec. 3. Coulomb friction on the contact surfaces was assumed:

$$\tau_k = f\sigma_n.$$

Here τ_k is the specific friction force (see Fig. 1), f is the friction coefficient, and σ_n is the normal stress on the contact surface. The friction coefficient f was varied in the calculations.

For the calculation, the following characteristics of the dislocation structure of St. 10 steel and the quantities included in the model were used: $\rho_{s0} = 1.3 \cdot 10^{10}$ cm^{-2}, $\lambda = 4.7 \cdot 10^{-4}$ cm, $G = 53{,}400$ MPa at $T = 850$°C, $\nu_{\text{D}} = 10^{12}$ sec^{-1}, $b = 3 \cdot 10^{-8}$ cm, $m = 3.1$, and $\dot{\varepsilon}_* = 9 \cdot 10^6$ sec^{-1}.

It is found that at rates of displacement of the movable die $v = 8.3 \cdot 10^{-1}$, $8.3 \cdot 10^{-2}$, and $8.3 \cdot 10^{-3}$ mm/sec and values of the friction coefficient $f = 0.05$, 0.07, and 0.10, respectively, the difference between the theoretical and experimental dependences $P(\Delta h)$ does not exceed 10% (Fig. 2). (The above-mentioned values of the friction coefficient are known in the theory and technology of PMW.)

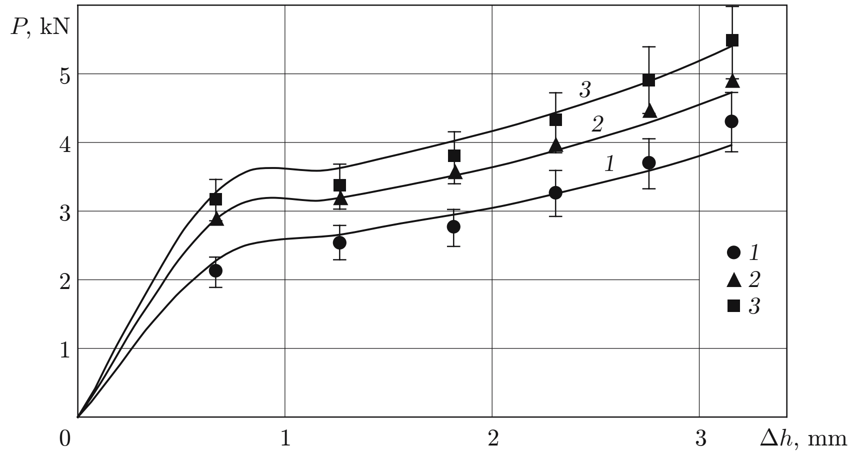

Fig. 2. Upsetting force versus displacement of the movable die at various rates of displacement of the movable die and values of the friction coefficient: 1) $v = 8.3 \cdot 10^{-3}$ mm/sec and $f = 0.05$; 2) $v = 8.3 \cdot 10^{-2}$ mm/sec and $f = 0.07$; 3) $v = 8.3 \cdot 10^{-1}$ mm/sec and $f = 0.10$; curves refer to the calculation and points to the experiment.

Conclusions. Based on a combinations of the basic micro- and macro-concepts and methods of micro- and macro-description of the plastic deformation of metals, a physicomathematical model of plasticity was proposed which, in contrast to the classical theory, takes into account loading history over a wide range of temperatures and strain rates and simplifies finite-element calculations of technological operations and processes of metal plastic working. In addition, in the solution of technological problems, the model allows one to determine accumulated dislocation density and, hence, to predict the structure and properties in various volumes of semifinished workpieces produced by plastic working metal.

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